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RECIPROCAL ARCS

BY

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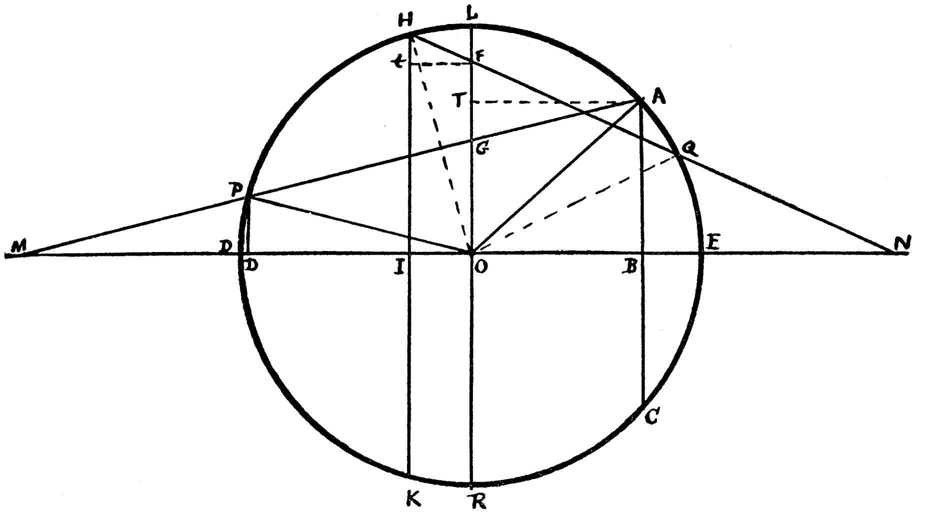

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The following dissertation on an old problem under a new name is respectfully and hopefully submitted to the mathematical world for criticism.

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M. H. B.

I propose to demonstrate the following proposition:

In a circle whose radius is 1, the cosine of one-half of a given arc is the square of the chord of one-third of another arc in the circle, and the cosine of one-half of the latter arc is the square of the chord of one-third of the former arc; or, to state it differently:

In a circle whose radius is 1, the distance from the center to the middle point of the chord of any arc, is the square of the chord of one-third of another arc in the circle; and the distance from the center to the middle point of the chord of that other arc, is the square of the chord of one-third of the former arc.

The proposition may be summarized as follows: Every arc has its reciprocal arc such that the cosine of one is the square of the chord of two-thirds of the other.

For convenience of construction and demonstration, the chords of the respective arcs are taken perpendicular to the horizontal diameter, so that the cosine of half the arc on one side of the center will be shown to be the square of the chord of one-third of the whole arc on the other side, and conversely.

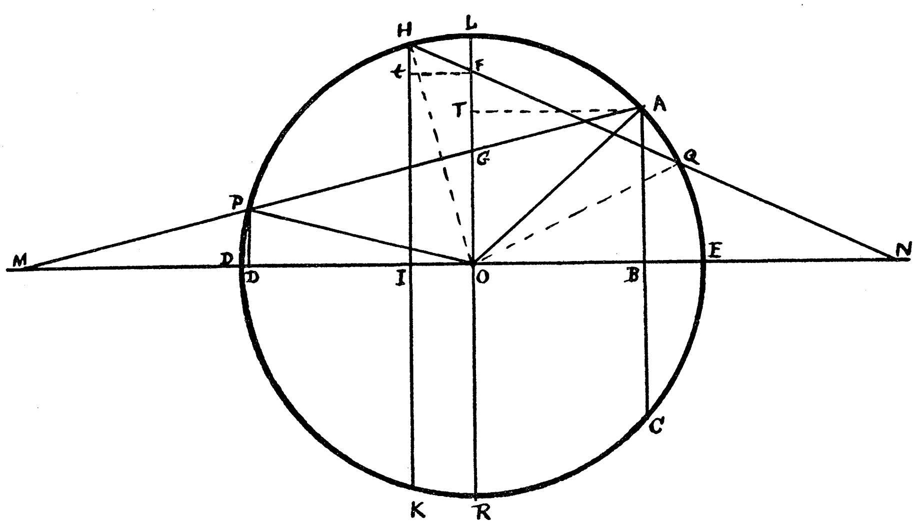

FIG. 1.

In the discussion, only the first named arc is given, but in the course of the construction the second named arc results, and from the cosine of half thereof the chord of one-third of the first named arc is determined.

Thus, in Fig. 1, let $HAECK$ be the circle, MN its horizontal diameter produced, AC the chord of the given arc, AE its half, OB the cosine of the arc AE. On DB as a diameter describe a circle cutting the perpendicular diameter at F. Let DO, the radius, equal 1. Then FO is a mean proportional between DO and OB. Hence $OB = \overline{FO}^2$. Call FO, y, and OB, y^2. Then $BE = 1 - y^2$.

A circle with F as a center and $BE = 1 - y^2$ as a radius will cut the main circle at H. From H drop HK perpendicular to DE, cutting it at I. On IE as a diameter describe a circle cutting the perpendicular diameter at G. GO will be a mean proportional between IO and OE, and $IO = \overline{GO}^2$. Call GO, x, and IO, x^2, then is OB the square of the chord of one-third of the arc HDK, and IO is the square of the chord of one-third of the arc AEC.

Proof: Draw Ft perpendicular to FO and HI, and draw AT perpendicular to FO and AB. As $IO = \overline{GO}^2 = x^2$, $DI = 1 - x^2$. Through A and G draw a line cutting the circle at P and the diameter DE produced, at M. Produce HF cutting the circle at Q and the diameter DE produced, at N.

$$AB = \tfrac{1}{2}AC = \sqrt{(1 - \overline{OB}^2)} = \sqrt{(1 - y^4)}.$$

$GA^2 = \overline{TA}^2 + \overline{TG}^2$; but $TA = OB = y^2$, and $TG = AB - GO$. Let $AB = c$. Then $GA = \sqrt{(y^4 + (c - x)^2)}$, and as $c^2 = 1 - y^4$,

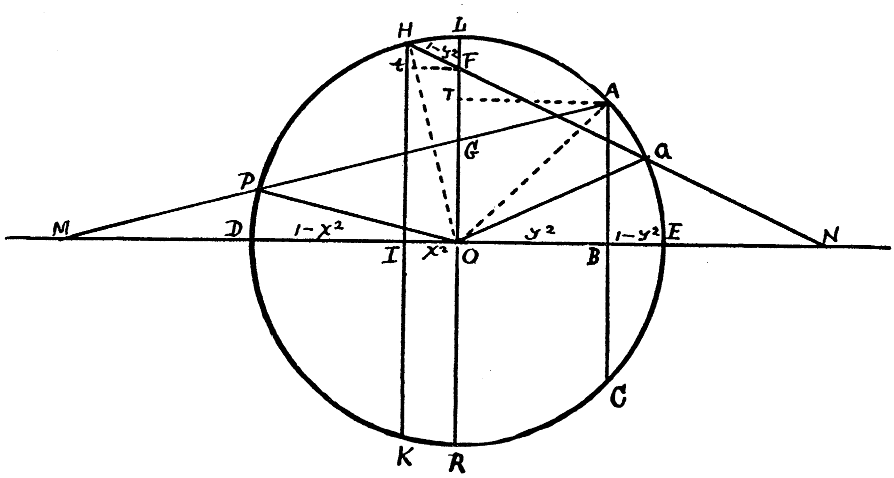

FIG. 2.

$$GA = \sqrt{(y^4 + 1 - y^4 + x^2 - 2cx)} = \sqrt{(x^2 - 2cx + 1)}.$$

(See Figure 2.)

For convenience, let $HI = k$, $FN = \alpha$ and $GM = a$. From similar triangles, FNO and HFT, we have $FN : FO :: 1 - y^2 : HI - FO = Ht$, or,

$$\alpha : y :: 1 - y^2 : k - y; \quad y - y^3 = \alpha(k - y); \quad \alpha = (y - y^3)/(k - y) \qquad (1)$$

$$(1 - y^2)^2 = \overline{Ft}^2 + \overline{Ht}^2 = (HI - tI)^2 + \overline{tF}^2 = (k - y)^2 + x^4.$$

As $tF = IO = x^2$, $x^4 = 1 - k^2$. Substituting in $(1 - y^2)^2 = (k - y)^2 + x^4$, we have

$$1 - 2y^2 + y^4 = y^2 - 2ky + 1; \quad y^4 - 3y^2 + 2ky = 0; \quad y^3 - 3y + 2k = 0,$$

which may be written

$$y - y^3 = 2k - 2y; \quad (y - y^3) \div (k - y) = 2.$$

We have shown that $(y - y^3) \div (k - y) = \alpha$, whence $\alpha = 2 = FN = DE$, the diameter of the circle. The circumscribing circle of the rightangled triangle FNO will have for a radius one-half of α or of 2, and a radius from the middle point of α to the vertex at O will $= 1$; but a line drawn to Q where α cuts the circle is equal to 1; hence FN is bisected by the circle at the point Q. Hence angle $QON = QNO$, and $FQO = 2QNO$. $HOD = OHQ + QNO = HQO + QNO$. But $QNO = \frac{1}{2}OQH$; whence $QNO =$ one-third of HOD and arc $QE =$ one-third of arc HD.

In the foregoing the arc HD is found as a result of the construction, but its trisection does not give any measure of the one-third of AE. If a should be found equal to α or if GA is equal to $1 - x^2$, the x equation would have the same form as the y equation, and without further reasoning the mind would conclude that the arc PD is equal to one-third of the arc AE.

Having found that $GA = \sqrt{(x^2 - 2cx + 1)}$ we have from similar triangles, GMO and GAT, the proportion:

$GA = \sqrt{(x^2 - 2cx + 1)} : a :: c - x : x; \quad x\sqrt{(x^2 - 2cx + 1)} = a(c - x).$

Squaring, we have:

$$x^4 - 2cx^3 + x^2 = a^2c^2 - 2a^2cx + a^2x^2 = 0,$$

$$x^4 - 2cx^3 + (1 - a^2)x^2 + 2a^2cx - a^2c^2 = 0.$$

Note.—That the only given value we have at this stage in the equation is c, and attention is called to the fact that we are concerned with the value of a only and not of x.

The different operations by which the value of a is arrived at will now be given.

In Fig. 1, LR and PA being chords intersecting at G, the product of the segments of one equals the product of the segments of the other. Let $PG = s$. We have $LG \cdot GR = AG \cdot PG$; $GO = x$, $LG = 1 - x$ and $GR = 1 + x$; AG has been shown to be $\sqrt{(x^2 - 2cx + 1)}$; whence $1 - x^2 = s\sqrt{(x^2 - 2cx + 1)}$; $\sqrt{(x^2 - 2cx + 1)} = (1 - x^2) \div s$, and $s = (1 - x^2) \div \sqrt{(x^2 - 2cx + 1)}$.

$MG = a$; $GO = x$; $a + \sqrt{(x^2 - 2cx + 1)} = a + (1 - x^2) \div s$ and $AB = c$. Whence, from similar triangles:

$$a : x : : a + (1 - x^2) \div s : c,$$
$$ac = ax + x(1 - x^2) \div s,$$
$$acs = asx + x - x^3,$$
$$x^3 - (1 + as)x + cas = 0.$$

Here we have a cubic having the x, c and a the same as in the biquadratic $x^4 - 2cx^3 + (1 - a^2)x^2 + 2a^2cx - a^2c^2 = 0$, and with the additional element s, which is a segment of a. Note also that the second power of x is wanting.

Taking the cubic as a divisor of the biquadratic, or as the biquadratic depressed to a cubic, and assuming the quotient, the linear, to be $x - d = 0$, and multiplying:

$$\begin{array}{lllll} x^3 & - (1 + as)x & = cas & = 0 & \\ x - d & & & = 0 & \\ \hline x^4 & & - (1 + as)x^2 & + casx & \\ & - dx^3 & & + d(1 + as)x & - casd \\ \hline \end{array}$$

we have: $x^4 - dx^3 - (1 + as)x^2 + (cas + d + das)x - casd = 0.$

Equating the coefficients of this with the corresponding coefficients of

$$x^4 - 2cx^2 + (1 - a^2)x^2 + 2a^2cx - a^2c^2 = 0,$$

we have $-d = -2c$; or $d = 2c$; $-1 - as = 1 - a^2$; $cas + d + das = 2a^2c$, and $casd = a^2c^2$. From $cas + d + das = 2a^2c$, after substituting the value of d, we have $cas + 2c + 2cas = 2a^2c$, and from $casd = a^2c^2$, after substituting the value of d, we have $2c^2as = a^2c^2$; $2s = a$, and $s = \frac{1}{2}a$. Substituting this value of s in $cas + 2c + 2cas = 2a^2c$, we have $\frac{3}{2}ca^2 + 2c = 2a^2c$; $2 = \frac{1}{2}a^2$; $a = 2$; whence $s = 1$ and $PM = 1$. Thus the hypotenuse of the rightangled triangle MGO is bisected by the circle at P.

The value of a is also found as follows: From the equations of the coefficients we have $-1 - as = 1 - a^2$; $a^2 - as - 2 = 0$; $a = \frac{1}{2}s + \sqrt{(\frac{1}{4}s^2 + 2)}$.

From

$$cas + 2c + 2cas = 2a^2c; \quad as + 2 + 2as = 2a^2; \quad 3as + 2 = 2a^2;$$
$$a^2 - 3as/2 - 1 = 0; \quad a = 3s/4 + \sqrt{(9s^2/16 + 1)}.$$

Placing the both values of a equal to each other, we have

$$3s/4 + \sqrt{(9s^2/16 + 1)} = \tfrac{1}{2}s + \sqrt{(\tfrac{1}{4}s^2 + 2)};$$
$$\frac{1}{4}s = \sqrt{\frac{1}{4}s^2 + 2} - \sqrt{\frac{9}{16}s^2 + 1}.$$

Squaring: $$\frac{s^2}{16} = \frac{s^2}{4} + 2 + \frac{9s^2}{16} + 1 - 2\sqrt{\left(\frac{s^2}{4}+2\right)\left(\frac{9s^2}{16}+1\right)};$$

$$\frac{12}{16}s^2 + 3 = 2\sqrt{\frac{9s^4}{64} + \frac{18s^2}{16} + \frac{s^2}{4} + 2};$$

$$\frac{3}{4}s^2 + 3 = 2\sqrt{\frac{9s^4}{64} + \frac{22s^2}{16} + 2}.$$

Squaring: $$\frac{9s^4}{16} + \frac{18s^2}{4} + 9 = \frac{9s^4}{16} + \frac{22s^2}{4} + 8;$$

$$1 = s^2; \quad s = 1.$$

Substitute value of s in $a^2 - as - 2 = 0$, $a^2 - a = 2$; $a = \frac{1}{2} + \sqrt{(\frac{1}{4} + 2)} = 2$. The negative value of the radical has not been considered.

If $a = 2$ and $s = 1$, the cubic $x^3 - (1 + as)x + cas = 0$, becomes $x^3 - 3x + 2c$, and the biquadratic becomes $x^4 - 2cx^3 - 3x^2 + 8cx - 4c^2 = 0$. This latter equation is divisible by the cubic $x^3 - 3x + 2c$, and the quotient is $x - 2c = 0$, as might be inferred from the linear $x - d = 0$ after d was found to equal $2c$.

Since we have shown that $a = 2$ and $s = 1$, we are now ready to close the demonstration of the original proposition. The radius $PO = 1$ by hypothesis, and PO, GP and PM being each equal to 1, the angle $GPO =$ the sum of the angles PMO and POM, and $PMO = POM$.

$$PAO = OPA = PMO + POM. \qquad AOE = PAO + PMO = 3PMO.$$

Hence, the arc $PD =$ one-third of the arc AE. Assuming, as we may, that D also represents the foot of a perpendicular from P to the diameter DE, and calling the perpendicular PD, we have the proportion $MP = 1 : PD :: MG = 2 : GO$, whence $PD = \frac{1}{2}GO = \frac{1}{2}x$; but PD is half of the chord of an arc twice PD or the chord of one-third of the arc AC. By analogy the chord from Q perpendicular to the diameter DE is the chord of one-third of the arc HDK and is equal to FO, and as $BO = FO$ squared and $IO = GO$ squared, the distance from O to the middle point of AC is the square of the chord of one-third of the arc HDK, and the distance from O to the middle point of the chord HK is the square of the chord of one-third of the arc AEC.

Q. E. D.

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